LIMITING EQUILIBRIUM OF A PLATE WEAKENED BY TWO CRACKS

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In the plane formulation of the problem of an infinite body weakened by two cracks of equal length (with a fair distance between them), a method of solution based on the dislocation description of the cracks was presented earlier by Smith [1]. The axially symmetrical case was considered by Ya. S. Uflyand [2], W. D. Collins [3], and Yu. N. Kuz'min [4]. The latter studied the equilibrium state of an unlimited space containing two coaxial slots of different radii. We shall now consider the analogous plane problem.

Let an infinite-isotropic-elastic plate be weakened by two open parallel cracks of unequal lengths (2a and 2b, a < b). The cracks lie at a distance h from each other and have a common symmetry axis. The x axis is directed along the smaller of the cracks, the y axis perpendicular to the plane of this crack in a direction opposite to the location of the other crack. Let us suppose that arbitrary normal breaking stresses $\sigma_a(x)$, $\sigma_b(x)$ and tangential stresses $\tau_a(x)$, $\tau_b(x)$ are specified on the surfaces of the cracks. There is no load at infinity. It is required to find the relationship between the load and the crack parameters if all the elastic constants of the plate material are known.

Thus the problem reduces to the integration of the elasticity-theory equations, subject to the boundary conditions

$$\begin{aligned} &\sigma_y = -\sigma_a(x), \quad \tau_{xy} = \tau_a(x) \quad (y = 0, \ |x| < a) \\ &\sigma_y = -\sigma_b(x) \quad \tau_{xy} = \tau_b(x) \quad (y = -h, \ |x| < b) \end{aligned} \tag{1}$$

In order to solve this problem, we shall use the Airy biharmonic function U(x, y) and its integrated cosine transform $G(\xi, y)$.

The stresses and elastic displacements are expressed in terms of $G(\xi, y)$ by the following equations [5]

$$\begin{gathered} \sigma_y = -\frac{2}{\pi}\int_0^\infty \xi^2 G(\xi, y)\cos\xi x\,d\xi, \qquad \tau_{xy} = \frac{2}{\pi}\int_0^\infty \xi\frac{\partial G}{\partial y}\sin\xi x\,d\xi \\ v = \frac{2(1+\nu)}{\pi E}\int_0^\infty \left[(1-\nu)\frac{\partial^3 G}{\partial y^3} - (2-\nu)\xi^2\frac{\partial G}{\partial y}\right]\frac{\cos\xi x}{\xi^2}\,d\xi \\ u = \frac{2(1+\nu)}{\pi E}\int_0^\infty \left[(1-\nu)\frac{\partial^2 G}{\partial y^2} + \nu\xi^2 G\right]\frac{\sin\xi x}{\xi}\,d\xi \end{gathered} \tag{2}$$

All space may naturally be divided into three regions $-\infty < y < -h$, $- < y < 0$, $0 < y < \infty$; subsequently the indices on the stress and strain components and on the functions U and G will indicate in which of these regions, the first, second, or third, the corresponding variable is defined. The stresses and strains must clearly be continuous in the separated regions. We thus arrive at the following boundary conditions:

for y = 0

$$\begin{gathered} \sigma_{1y} = \sigma_{2y}, \quad \tau_{1xy} = \tau_{2xy}, \quad \sigma_{1y} = -\sigma_a(x) \quad (|x| < a) \\ \tau_{1y} = \tau_a(x) \quad (|x| < a), \quad u_1 = u_2, \quad v_1 = v_2 \quad (|x| > a) \end{gathered}$$

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$$\sigma_{2y} = \sigma_{3y}, \quad \tau_{2xy} = \tau_{3xy}, \quad \sigma_{3y} = -\sigma_b(x) \quad (|x| < b)$$

$$\tau_{3xy} = \tau_b(x) \quad (|x| < b), \qquad u_2 = u_3, \quad v_2 = v_3 \quad (|x| > b)$$

We take the Airy functions in the corresponding region as

$$\begin{aligned} U_1(x, y) &= \frac{2}{\pi} \int_0^\infty [(A + B\xi y) e^{-\xi y} + (K + L\xi y) e^{\xi y}] \cos \xi x d\xi \\ U_2(x, y) &= \frac{2}{\pi} \int_0^\infty [(C + E\xi y) \operatorname{sh} \xi y + (D + F\xi y) \operatorname{ch} \xi y] \cos x\xi d\xi \\ U_3(x, y) &= \frac{2}{\pi} \int_0^\infty [(R + H\xi y) e^{-\xi y} + (M + N\xi y) e^{\xi y}] \cos \xi x d\xi \end{aligned} \tag{4}$$

On the basis of the behavior of the stress and strain components at infinity we put $K(\xi) = L(\xi) = R(\xi) = H(\xi) = 0$. The eight remaining unknown functions are determined from the boundary conditions (3), which, on allowing for (2) and (4), lead to a system of four paired integral equations

$$\begin{aligned} &\frac{2}{\pi} \int_0^\infty \xi^2 A \cos \xi x \, d\xi = \sigma_a(x) \\ & \qquad\qquad\qquad\qquad (0 < x < a) \\ &\frac{2}{\pi} \int_0^\infty \xi^2 (B - A) \sin \xi x \, d\xi = \tau_a(x) \\ &\frac{2}{\pi} \int_0^\infty \xi^2 (M - N\mu) e^{-\mu} \cos \xi x d\xi = \sigma_b(x) \\ & \qquad\qquad\qquad\qquad (0 < x < b) \\ &\frac{2}{\pi} \int_0^\infty \xi^2 [M + N(1 - \mu)] e^{-\mu} \sin \xi x d\xi = \tau_b(x) \\ &\int_0^\infty \xi (B + E) \sin \xi x d\xi = 0, \quad \int_0^\infty \xi (B - F) \cos \xi x d\xi = \sigma \qquad (a < x < \infty) \\ &\int_0^\infty (F \operatorname{sh} \mu - E \operatorname{ch} \mu + Ne^{-\mu}) \xi \sin \xi x d\xi = 0 \qquad (b < x < \infty) \\ &\int_0^\infty (F \operatorname{ch} \mu - E \operatorname{sh} \mu - Ne^{-\mu}) \xi \cos \xi x d\xi = 0 \\ & \qquad\qquad (\mu = \xi h) \end{aligned} \tag{5}$$

We shall base our considerations on a simpler system of equations obtained by preliminary integration of the first and third equations from 0 to x and differentiation of the fifth and seventh with respect to x:

$$\begin{aligned} &\int_0^\infty \lambda A \sin \lambda x_1 d\lambda = \frac{\pi}{2} a^2 \int_0^{x_1} \sigma_a(ax_1) dx_1 = 2G_a(x_1) \\ &\int_0^\infty \lambda^2 (B - A) \sin \lambda x_1 d\lambda = a^3 \tau_a(ax_1) = 2Q_a(x_1) \qquad (0 < x_1 < 1) \\ &\int_0^\infty \lambda (M - N\eta) e^{-\eta} \sin \lambda x_1 d\lambda = \frac{\pi}{2} b^2 \int_0^\infty \sigma_b(bx_1) dx_1 = 2G_b(x_1) \\ &\int_0^\infty \lambda^2 [M + N(1 - \eta)] e^{-\eta} \sin \lambda x_1 d\lambda = b^3 \tau_b(bx_1) = 2Q_b(x_1) \\ &\int_0^\infty \lambda^2 (B + E) \cos \lambda x_1 d\lambda = 0, \qquad \int_0^\infty \lambda (B - F) \cos \lambda x_1 d\lambda = 0 \\ &\int_0^\infty \lambda^2 (F \operatorname{sh} \eta - E \operatorname{ch} \eta + Ne^{-\eta}) \cos \lambda x_1 d\lambda = 0 \qquad (1 < x_1 < \infty) \\ &\int_0^\infty \lambda (F \operatorname{ch} \eta - E \operatorname{sh} \eta - Ne^{-\eta}) \cos \lambda x_1 d\lambda = 0 \\ &(\xi = \lambda / a = \lambda / b, \ x = ax_1 = bx_1, \ h = b\delta, \ \eta = \lambda\delta) \end{aligned} \tag{6}$$

(Subsequently the index on x is omitted.)

The integral substitution [6]

$$B+E=\frac{1}{\lambda^2}\int_0^1 f'(t)J_0(\lambda t)\,dt,\qquad B-F=\frac{1}{\lambda}\int_0^1 \varphi'(t)J_0(\lambda t)\,dt$$
$$F\,\mathrm{sh}\,\eta+Ne^{-\eta}-E\,\mathrm{ch}\,\eta=\frac{1}{\lambda^2}\int_0^1 \psi'(t)J_0(\lambda t)\,dt \tag{7}$$
$$F\,\mathrm{ch}\,\eta-Ne^{-\eta}-E\,\mathrm{sh}\,\eta=\frac{1}{\lambda}\int_0^1 \zeta'(t)J_0(\lambda t)\,dt$$

reduces the system of integral equations (6) to a system of four Fredholm equations of the second kind

$$\Phi(x)+\frac{2}{\pi}\int_0^1 K_1(x,v,\delta)\Psi(v)\,dv+\frac{2}{\pi}\int_0^1 K_2(x,v,\delta)Z(v)\,dv=G_a(x)-\psi(1)A_1-\xi(1)A_2$$
$$F(x)-\frac{2}{\pi}\int_0^1 K_3(x,v,\delta)\Psi(v)\,dv+\frac{2}{\pi}\int_0^1 K_4(x,v,\delta)Z(v)\,dv=Q_a(x)-\psi(1)A_3-\xi(1)A_4$$
$$Z(x)+\frac{2}{\pi}\int_0^1 K_5(x,v,\delta)F(v)\,dv+\frac{2}{\pi}\int_0^1 K_6(x,v,\delta)\Phi(v)\,dv=G_b(x)-f(1)A_5-\varphi(1)A_6 \tag{8}$$
$$-\Psi(x)+\frac{2}{\pi}\int_v^1 K_7(x,v,\delta)F(v)\,dv+\frac{2}{\pi}\int_0^1 K_8(x,v,\delta)\Phi(v)\,dv=Q_b(x)-f(1)A_7-\varphi(1)A_8$$

Here

$$K_i=\frac{2}{\pi}v\int_0^1\frac{dt}{t\sqrt{t^2-v^2}}\int_0^t\sqrt{t^2-u^2}\,du\int_0^\infty\lambda^2\varphi_i(\lambda\delta)\sin\lambda x\cos\lambda u\,d\lambda$$
$$A_i(x,\delta)=\frac{2}{\pi}\int_0^1\frac{du}{\sqrt{1-u^2}}\int_0^\infty\varphi_i(\lambda\delta)\sin\lambda x\cos\lambda u\,d\lambda\qquad(i=1,\ldots,8)$$
$$\varphi_1=\varphi_5=-\delta e^{-\eta},\qquad \varphi_2=\varphi_6=e^{-\eta}(1+\eta)$$
$$\varphi_3=-\varphi_7=e^{-\eta}(\eta-1),\qquad \varphi_4=-\varphi_8=-\lambda\eta e^{-\eta}$$

The functions $f(t)$, $\varphi(t)$, $\psi(t)$, $\zeta(t)$ are associated with $F(v)$, $\Phi(v)$, $\Psi(v)$, $Z(v)$ by the relations

$$f(t)=\frac{2}{\pi}\int_0^t\frac{vF(v)\,dv}{\sqrt{t^2-v^2}},\qquad \varphi(t)=\frac{2}{\pi}\int_0^t\frac{v\Phi(v)\,dv}{\sqrt{t^2-v^2}}$$
$$\psi(t)=\frac{2}{\pi}\int_0^t\frac{v\Psi(v)\,dv}{\sqrt{t^2-v^2}},\qquad \zeta(t)=\frac{2}{\pi}\int_0^t\frac{vZ(v)\,dv}{\sqrt{t^2-v^2}} \tag{9}$$

Thus, if we can find the solution of the system (8), the stress/strain state of a strip weakened by two cracks is given by (9), (7), (4), and (2). In order to determine a number of important characteristics, there is no need, as is often the case when using the method of paired equations, to calculate all the coefficients in Eqs. (4), since these characteristics may easily be expressed in terms of the solutions of the Fredholm equations (8). For example, the normal displacements of the points on the surfaces of the cracks (determining their width) take the form

$$(v_1-v_2)|_{y=0}=\frac{4(1-\nu^2)}{\pi Ea}\int_x^1\frac{\varphi'(t)\,dt}{\sqrt{t^2-x^2}},\quad (v_2-v_3)|_{y=-h}=\frac{4(1-\nu^2)}{\pi Eb}\int_x^1\frac{\zeta'(t)\,dt}{\sqrt{t^2-x^2}}$$

If the parameter $\delta^{-1}=b/h$ is small (the case in which the distance between the slits is large compared with their dimensions), the solution of the system (8) may be carried out by expansion in series with respect to this parameter. Calculations carried out for the case $\sigma_a(x)=\sigma_b(x)=P$ and $\tau_a(x)=\tau_b(x)=0$ gave the following equations for the displacements of the points on the surfaces of the cracks:

$$(v_1-v_2)|_{h=0}=\frac{4Pa(1-\nu^2)}{E}\sqrt{1-x^2}\left[\Delta_1(x,\delta)-\frac{a^2}{b^2}\nabla_1(x,\delta)\right]$$
$$(v_2-v_3)|_{y=-h}=\frac{4Pb(1-\nu^2)}{E}\sqrt{1-x^2}\left[\Delta_2(x,\delta)-\frac{b^2}{a^2}\nabla_2(x,\delta)\right]$$

where

$$\Delta_i(x,\delta) = 1 - c_1^i\delta^{-4} - \delta^{-6}[(1/3 + 2/3x^2)c_2^i + c_3^i] + \ldots$$

$$\nabla_i(x,\delta) = c_4^i\delta^{-2} - [c_5^i(1/3 + 2/3x^2) + c_6^i]\delta^{-4} - [c_7^i(1/5 + 4/5x^2) + 8/15x^4) + c_8^i(1/3 + 2/3x^2) - c_9^i]\delta^{-6} + \ldots \quad (i = 1, 2)$$

$$c_1^1 = 3.75,\quad c_2^1 = 2.25,\quad c_3^1 = 7.875,\quad c_4^1 = 1.5,\quad c_5^1 = 3.75$$
$$c_1^2 = 3.75,\quad c_2^2 = 7.875,\quad c_3^2 = 13.5,\quad c_4^2 = 1.5,\quad c_5^2 = 3.75$$
$$c_6^1 = 1.875,\quad c_7^1 = 6.5,\quad c_8^1 = 6.5,\quad c_9^1 = 13.425$$
$$c_6^2 = 1.875,\quad c_7^2 = 6.5,\quad c_8^2 = 13.125,\quad c_9^2 = 3.438$$

The solution of the problem was carried out without allowing for the forces of molecular cohesion. In order to find the lengths $2a$ and 2b of the equilibrium cracks, we require that their opposite surfaces should close smoothly toward the ends. These lengths may be determined from relationships derived earlier [7] and written in dimensionless form [8]

$$\lim_{x\to1}\frac{d}{dx}\left[\frac{1}{2}(v_1 - v_2)|_{y=0}\right]\sqrt{1-x} = \frac{2K(1-\nu^2)\sqrt{a}}{\pi E}$$
$$\lim_{x\to1}\frac{d}{dx}\left[\frac{1}{2}(v_2 - v_3)|_{y=-h}\right]\sqrt{1-x} = \frac{2K(1-\nu^2)\sqrt{b}}{\pi E}$$

Calculation of the limits gives

$$\frac{P\sqrt{a}}{\sqrt{2}}\Delta_3(1,\delta) = \frac{K}{\pi},\qquad \frac{P\sqrt{b}}{\sqrt{2}}\Delta_4(1,\delta) = \frac{K}{\pi}$$
$$\Delta_3(1,\delta) = 1 - \frac{a^2}{b^2}1.5\delta^{-2} - \left(3.75 - \frac{a^2}{b^2}5.625\right)\delta^{-4} - \left(10.125 - \frac{a^2}{b^2}2.945\right)\delta^{-6} + \ldots \tag{10}$$
$$\Delta_4(1,\delta) = 1 - \frac{b^2}{a^2}1.5\delta^{-2} - \left(3.75 - \frac{b^2}{a^2}5.625\right)\delta^{-4} - \left(21.375 - \frac{b^2}{a^2}19.557\right)\delta^{-6} + \ldots$$

It follows from Eq. (10) that for a fixed distance between the cracks, the large crack starts propagating load P, and owing to the instability of the dynamic equilibrium of the cracks for uniformly distributed loading of the body at infinity it becomes an arterial crack.

For $a = b$ the results of the present investigation coincide with Smith's [1]. In the limit, as $h \to \infty$, Eqs. (10) take a form coinciding with the known solution for an isolated Griffith crack in a homogeneous body.

It should be noted that our present hypothesis to the effect that the cracks only propagate in the planes in which they are situated will undoubtedly give overestimated breaking stresses.

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